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Properties of upsilon meson based on the two intertwined spaces

Scientific research paper

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ABSTRACT

This paper studies the important concepts in the fundamental optimization of mass and thermal properties of strong interactions based on quantum approximate operators. It explores and analytically calculates the radial part of the Schrödinger equation at finite temperature using the two intertwined spaces based on the Wick ordering method in the Bernoulli Potential. We provide analytical expressions for the ground state energy eigenvalues to define the zeroth approximation. We use a combination of the mathematical series terms (Bernoulli Serie) with the spins dependent part on the modified potential. Bernoulli potential refers to some of the potential types that are returned to the Bernoulli Serie. The method presented in this study allows us to rewrite some of the exponential potentials simply to even power levels in the Bernoulli series. Given that the Hulthen potential is a modified form of main hadronic interaction potentials, it is used to calculate the bound state mass and properties of hadronic atoms such as π-atoms, κ-atoms or various hadronic structures with positive/negative and heavy/light quark bound states such as ${D_s}^+, {B_c}^+, B^+, D^+, D^{*+}$ at finite temperatures. The main goal of this research is on the combined quantum operators and two intertwined spaces within the Bernoulli expansion to determine the best approximation of upsilon meson mass and thermal properties.

1 Introduction

In recent years, theoretical physicists have become increasingly interested in the study of the mass and thermal properties of hadronic-bound states [1]. This is because it is necessary to investigate fundamental and experimental data and most efficiently, it is made feasible by the radial part of the Schrödinger equation, which provides the necessary details to characterize the bound system of elementary particles under new investigation. The study of bound states of two, three, and more quarks exacerbations, in a classical limit and a high-temperature environment within the infinite temperature, is important for understanding the behavior of exotic bound states of quarks near the deconfinement temperature in high energy interacting media and environments. The behavior of these states near the deconfinement temperature is incalculable and unclear, and various models present and estimate the

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approximation mass spectra and eigenenergy values by increasing, or decreasing with temperature increments. In addition, the color screening radius (Deby mass, Debye-Hückel length), below which binding becomes impossible, plays a fundamental contribution in determining the properties of these hadronic states at finite temperatures [2]. In this research, we try to define the interactions of the upsilon meson-bound state which is like $D_s^+, B_c^+, B^+, D^+, D^{*+}$bound systems. Potential interaction of upsilon meson bound states which has to comprise Coulomb and confining terms, is presented in the form of Bernoulli series while the main fundamental quantum particle behavior gives the harmonic oscillator potential [3-5].

These internal-external potentials play a key role in the upsilon meson coupled state, including the mixture of needing properties, which have been extensively discussed in the literature. It is worth noting that the development of new potential models is crucial for understanding the properties of hadronic bound states and other quantum systems. It is indeed true that the behavior of selective potentials in quantum field theory and quantum mechanics has been very successful in describing the meson-bound state data at finite temperatures. However, to take into account the relativistic effects, it is necessary to consider relativistic mass and spin-dependent potentials in the Schrödinger equation. The exact solubility of some potentials within the Schrödinger equation is of great importance as it allows analytical calculation of all spectra and energy eigenvalue of radial and orbital excited states. While some potentials can be solved exactly, others require approximation or numerical methods. For instance, the Bernoulli potential which consists of exponential-type potentials, has been investigated here. Furthermore, we have mentioned the quantum harmonic oscillator as a main unique internal interaction, is an important application and model when describing the bound state hadronic particles. As we know, harmonic oscillator eigenvalue problems can be solved analytically when the exact solution of a problem cannot be found, it is appreciable to use approximation methods such as perturbation theory. The perturbation theory approach has been adopted in several ways to calculate and determine the energy eigenvalues of the ground and excites states of coupled particles. By using a harmonic oscillator method and boundary conditions of the bound state, the quano-mechanical properties of the bound state such as mass and thermal characteristics can be solved. Therefore, these perturbed harmonic oscillators may be calculated using computational and analytical methods with theoretical contributions. In this article, we use the two intertwined spaces based on the Wick ordering method among other approaches, which has significant contributions to approximating and developing mathematical techniques for finding the eigenvalues and eigenfunctions of quantum systems with harmonic oscillator main potential and external potential in recent years. As we know the Wick ordering method is a useful analytical method for solving and approximating the radial Schrödinger equation in quantum mechanics. It is based on the idea of separating the potential into a real part and an imaginary part, which is then treated separately. The real part of the potential can be solved exactly, while the imaginary part can be treated as a perturbation to the real part. This method has been successfully applied to study hadronic-bound systems in various conditions, including the upsilon meson-bound state in this research. Also, in the analysis of the characteristic of the bound states, a transformation from one space to another space is considered to obtain the answer. These intertwined spaces based on the Wick ordering method are a useful analytical method for solving and approximating the radial Schrödinger equation, and it has been successfully applied to study hadronic bound systems in various conditions. In particular, this method can be used to analytically resolve the multidimensional n-dimensional radial part of the Schrödinger equation for the real part of the potential, and study upsilon meson bound state or quarkoniums dissociation in different states and different media such as finite temperature environment. The two intertwined spaces based on the Wick ordering method in the presence effect of external electric and magnetic fields have also been applied to solve the radial Schrödinger equation. The purpose of this research is to use the perturbation and approximate solution method to calculate the zero-energy correction and obtain the generalized energy eigenvalues for the quantum harmonic oscillator with Bernoulli potential. It explores the radial Schrödinger equation at exact temperature applying and implementing the transformation and intertwined two spaces method for potential part. We provide analytical expressions for the energy eigenvalues and mass spectrum. The obtained results for upsilon meson agree with current experimental data for different quantum numbers [6,7].

The numerical analysis indicates a distinct behavior for different quantum numbers concerning temperature dependence. Our temperature-dependent results for the ground states are determined at temperatures $T = 30, 70, 90\ (MeV)$.

The remainder of this research is laid out in the following manner: Section 2, introduces the Bernoulli generating function and Bernoulli potential. Section 3, introduces the Sturmian representation where two intertwined spaces are presented. In Section 4, the Wick ordering method is described. In Section 5, the mass spectrum of upsilon meson, with relativistic corrections at finite temperature is defined. In section 6, the numerical calculation is shown. Finally, in Section 7, concluding remarks are provided.

2 The Bernoulli generating function

The basis of the mathematical concept of Bernoulli potential under the Bernoulli numbers in 1755 by Leonhard Euler was established. Euler, based on the concept of a generating function described the Bernoulli numbers as a coefficient of the function $B(r) = \frac{r}{e^{r-1}} = \sum_{i=0}^{\infty} \frac{B_i r^i}{i!}$ [8,9]. If we suppose two series, first or main: $f(r) = \sum_{i=0}^{\infty} a_i r^i$ with coefficient $\{a_i\}$ and the second series: $g(r) = \sum_{i=0}^{\infty} b_i \frac{r^i}{i!}$, with the coefficient $\{b_i\}$. Then the exponential functions: can be generated by the coefficient $i!\{a_i\} = \{n!\, a_i\}$. The function that is the main generating function $f(r)$ is the exponential generating function of $g(x)$. Therefore, the function $B(r) = \frac{r}{e^{r-1}}$ based on the Maclaurin series expansion can be calculated by the i's order derivatives

$$\begin{aligned} B(r) &= \frac{r}{e^{r-1}} = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{d^i(r^i)}{dr^i} \\ &= B_0 + B_1 x + B_2 \frac{r^2}{2!} + +B_3 \frac{r^3}{3!} + B_4 \frac{r^4}{4!} + \cdots \\ &= 1 + \left(-\frac{1}{2}\right) x + \left(\frac{1}{6}\right) \frac{x^2}{2!} + \left(-\frac{1}{30}\right) \frac{x^4}{4!} \\ &+ \left(\frac{1}{42}\right) \frac{x^6}{6!} + \cdots. \end{aligned} \tag{1}$$

The coefficient of this series presents the Bernoulli numbers and equivalents to

$B_0 = 1$, $B_1 = -1/2$, $B_3 = 0$, $B_4 = -1/30$, $B_5 = 0$,

$B_6 = 1/4$, $B_7 = 0$, $B_8 = -1/30$, $\cdots$ and all the Bernoulli numbers have properties:

1- B_i is a rational number.
2- B_{2i} has alternates sign of $i \geq 1$ then $B_{4i} < 0$ and $B_{4i+2} > 0$.
3- $B_{2i+1} = 0$, for all $i \geq 1$.
4- $|B_{2i}|$ increases very quickly.
5- $cot\,(r) = \sum_{i=0}^{\infty} (-1)^i \frac{2B_{2i}(2r)^{2i-1}}{(2i)!}$ for $|r| < \pi$.
6- All the odd Bernoulli numbers are zero except B_1.

Each exponential potential that is transforming to the Bernoulli series can be used to describe bound state interactions by greater orders of r. As it is known, odd powers greater than one are zero, and in this way, the approximation with the Bernoulli series with even degrees will be determined. This subject is very significant for approximation in interaction potentials and calculation of eigenvalues and mass spectrum in Schrödinger's equation, which is discussed in this article.

3 Sturmian representation for two intertwined spaces

We present an expansion for the coupled-body Schrödinger problem. It can describe the bound state properties of particles based on the Sturmian representation**.** As we know, a specific set of eigenfunctions of the radial Schrödinger equation, known as the Sturmian function, have proven to be a valuable tool in addressing certain coupled body problems related to the radial Schrödinger wave functions.

The Sturmian function $S_{nl}(r)$:

$$\left[\begin{matrix} -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{l(l+1)\hbar^2}{2\mu r^2} + \\ +\alpha_{nl} U - E_0 \end{matrix} \right] S_{nl}(r) = 0. \tag{2}$$

is the solution of the radial part of the Schrödinger equation [10]. These functions have a notable benefit over Schrödinger functions when used as a basis for expansion, since they create a complete set that is not continuous, regardless of the potential between particles. When choosing a different expansion basis of intertwined spaces for the coupled-body state, three key conditions must be satisfied. Firstly, the series must approach convergence at a reasonable rate. Secondly, the continuum and its inherent complexity must be avoided to justify the use of the new series over a usual eigenvector and eigenfunction series. Lastly, the boundary constraints set and restrictions imposed by the new functions must be elementary and uncomplicated. For instance, quantum harmonic oscillator wave functions satisfy the second condition and requirement, but at the expense of losing and overburdening the simplicity of their asymptotic behavior of boundary conditions, which makes synthesizing an outgoing spherical symmetric wave difficult. However, we have discovered a set of functions that satisfies the second and third requirements. Rosner, Quigg, and Gazeau were dealing with the same topic. They have drawn attention to the fact that the origin of the Sturmian representation and changing the independent coordinate can describe coupled-body problems related to the radial Schrödinger equation. The mechanism of transforming the independent parameter has long been a useful strategy to solve the radial Schrödinger equation with different type potentials like $U(r) \approx \sum r^{\alpha}$ power-law potential especially with the bound state interaction in a central potential. The power-law potential with a variable exponent is a flexible resource that can be utilized to analyze the actions of systems where particles interact with one another, and its usefulness extends to various areas and branches of physics, including condensed matter physics, astrophysics, and fundamental particle physics. Here are some specific examples of its applications. This type of potential has been used to study the behavior of quarks and gluons in the quark-gluon plasma, and hadronic strong interaction which is a state of matter that existed in the early universe and can be recreated in particle accelerators. In this article, the study of coupled-body involves the use of the Bernoulli potential, and this revision clarifies that the Bernoulli potential was useful in the study of coupled-body upsilon meson. Based on Section 1, the subject of the paper is referred to as the Bernoulli potential which is a power-law potential with a variable exponent ($\alpha \geq 1$). The bound state issues can be naturally explained using the origin of the Sturmian representation and the equivalence transformed space for coupled-body. We have investigated this concept further within the framework of exponential Bernoulli potentials, which can be generalized to potentials with multiple powers of r, as

$$U_B(r) = B_0 + B_1 r + B_2 \frac{r^2}{2!} + B_3 \frac{r^3}{3!} + B_4 \frac{r^4}{4!} + \cdots. \qquad (3)$$

In cases where exact solutions for exponents $\sum r^{\alpha}$ are not available, a more general equivalence emerges as a change of variable in the wave function

$$R(r) \approx e^{-a(r)} \approx e^{-r^{1+\sigma}}, r = (cq)^{\beta},\ \beta = 2\rho = 2\left(\frac{1}{1+\sigma}\right),\ \sigma \geq 0, \qquad (4)$$

which can map the nonrelativistic radial Schrödinger equation and its solutions for $V(r) \approx \sum r^{\alpha}$ potential types. Parameter c is constant. For different values of α, the coupled system has to create bound states, with boundary conditions on the wave functions being linked by this transformation. If we focus on long distances limit and using analytical methods, we can typically determine the asymptotic properties and long-term behavior of the wave function

$$R\left(r\left((cq)^{\beta}\right)\right) \approx e^{-a\left(r\left((cq)^{\beta}\right)\right)} \approx e^{-(cq)^{2\left(\frac{1}{1+\sigma}\right)}}, \qquad (5)$$

for $\boldsymbol{r} \to \infty$, where $\boldsymbol{a}(\boldsymbol{r})$ can be obtained for certain classes of potentials. For the large distances potentials such as Coulomb or Yukawa-type potentials ($\alpha \leq 0$), $\sigma = 0$, for external harmonic potential ($\alpha = 2$), $\sigma = 0$, anharmonic potential ($\alpha > 2$), $1 < \sigma \leq 2$ and Cornell potential $\sigma = 1$. From the previous references to physics, we have realized that the concepts of harmonic oscillators hold significant relevance across various areas of physics, with the former being a crucial tool for modeling physical systems. The harmonic oscillator analogy is extensively employed in attempts to solve quantum mechanical problems, as many physical scenarios can be mapped onto a harmonic oscillator with appropriate boundary conditions. This stems from the fact that the harmonic oscillator eigenvalue problem has an analytic solution, thus allowing more accurate results and better approximation for solutions. We note that one has studied and analyzed potentials with an adjustable number of power factors of $\sum r^{\alpha}$, and has used a variable change to interpret the Sturmian equation and two intertwined spaces in the context of conventional physical radial Schrödinger equations. We also mention that the bound states with a based-on quantum harmonic oscillator behavior (principal harmonic potential) can be tackled using algebraic methods, such as the Wick ordering method. In the following paragraphs, our examination begins with the radial Schrödinger equation that applies to space with N-dimensions.

4 Radial Schrödinger equation in two intertwined spaces based on Wick ordering

In this study, theoretically and approximately, the bound state system solution and eigenvalues answered by the n-dimension radial Schrödinger equation in two intertwined spaces based on Wick ordering has been described. For this opinion, the two intertwined spaces based on the Wick ordering method are used. We have to predict the relativistic mass spectrum using the mechanism of intertwined spaces based on normal order (when all raising operators are to the left of all lowering operators) that is described in this paragraph and the quantum field theory method that will be presented in the next paragraph. Our start point is the radial Schrödinger equation in n-dimension space which describes the interaction of two particles with the masses m_i, m_j in the potential $U_{ij}(r)$ that lead us to create the stable bound state:

$$\left(\left\{-\frac{\hbar^2}{2m_i}\frac{d}{dr}r^{n-1}\frac{d}{dr}\right\} + \left\{-\frac{\hbar^2}{2m_j}r^{1-n}\frac{d}{dr}r^{n-1}\frac{d}{dr}\right\} + \frac{\hbar^2\ell(\ell+n-2)}{2m_i r^2} + \frac{\hbar^2\ell(\ell+n-2)}{2m_j r^2} + U_{ij}(r) - -E(m_i,m_j)\right)R(r) = 0, \tag{6}$$

then

$$\left(\frac{\hbar^2}{2\mu}\Delta + \frac{\hbar^2\ell(\ell+n-2)}{2\mu} - W(U,\mathrm{E})\right)R(r) = 0. \tag{7}$$

Equation (7), based on the Laplacian operator effect functions $R(r)$ and $\left(r^{\frac{1-n}{2}}R(r)\right)$ in n-dimension space [11]

$$\Delta = \frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr} - \frac{\ell(\ell+n-2)}{r^2},$$

$$\Delta R(r) = \Delta\left(r^{\frac{1-n}{2}}R(r)\right). \tag{8}$$

One can present Eq. (7) as follows

$$\mathfrak{R}'' - \frac{L(L+1)}{r^2}\mathfrak{R} + \frac{W(U,\mathrm{E})}{r^2}\mathfrak{R} = 0, \tag{9}$$

where $\mathfrak{R}(r) = r^{\frac{1-n}{2}}R(r)$, $\frac{1}{\mu} = \frac{1}{m_i} + \frac{1}{m_j}$ and μ is the reduced mass, ℓ is the angular momentum quantum number, L is a parameter that can be as a new auxiliary space i.e., we define the radial Schrödinger equation in the new L-dimension space which is linked to the n-dimension space $L = \frac{2\ell+n-3}{2}$, and $L(L+1) = \frac{4\ell^2+4\ell(n-2)+n^2-4n+3}{4}$. As we introduced in paragraph 2, Eq. (5), by changing [11, 12]

$$r = (cq)^{2\rho}, \mathfrak{R}(r) \to \mathfrak{R}((cq)^{2\rho}), \tag{10}$$

this means that always maps $r = 0$ into $q = 0$ and maps $r = \infty$ into $q = \infty$, and c-is recalling constant. Based on two intertwined spaces is transformed by relations

$$\frac{d}{dr} = \frac{1}{\rho c^\rho}q^{1-\rho}\frac{d}{dq}$$

$$\frac{d^2}{dr^2} = \left(\frac{1}{\rho c^\rho}q^{1-\rho}\frac{d}{dq}\right)\left(\frac{1}{\rho c^\rho}q^{1-\rho}\frac{d}{dq}\right) =$$

$$\frac{1}{(\rho c^\rho)^2}q^{1-\rho}\left((1-\rho)q^{-\rho}\frac{d}{dq} + q^{1-\rho}\frac{d^2}{dq^2}\right)$$

Hence, the radial Laplacian in an n-dimensional Riemannian space is

$$\varDelta_r = \frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr} \to$$

$$\varDelta_q = \frac{d^2}{dq^2} + \frac{\mathcal{D}-1}{q}\frac{d}{dq},$$

and then Eq. (8) reads

$$\left(\frac{d^2}{dq^2}-\frac{1-2\rho+n\rho}{q}\frac{d}{dq}-\frac{\ell(\ell+n-2)\rho^2}{q^2}+4\rho^2c^{4\mu}q^{4\mu-2}W((cq)^{2\rho})\right)\Re((cq)^{2\rho})=0, \quad (11)$$

$D=2-4\rho+2n\rho$ and then we supposed c is a real and positive number in this article and equivalent to 1. After some calculations like Eq. (8) with new axillary D-dimension space $\Re((q)^{2\rho})\to q^{\frac{1-D}{2}}\Phi$, we define

$$\Phi''-\frac{L_D(L_D+1)-\ell(\ell+n-2)\rho^2}{q^2}\Phi+4\rho^2q^{4\mu}W((q)^{2\rho})\Phi=0$$

where

$$L_D=\frac{4\ell\rho+D-3}{2},$$

$$L_D(L_D+1)=\frac{16\ell^2\rho^2+16\ell\rho^2(n-2)+(D-2)^2-1.}{4}$$

5 Coupled-state in QFT

The total energy and mass of bound states can be effectively determined within nonrelativistic quantum mechanics when an appropriate interaction potential is selected. Despite this, the nonrelativistic radial Schrödinger equation $\hat{H}R(r)=E_{n\ell}(\mu)R(r)$ which provides and employs mathematical techniques for accurate description of bound states is no longer sufficient for interpreting contemporary experimental results from hadronic physics, as relativistic corrections must be considered. Despite this, the nonrelativistic Schrödinger equation remains a dependable instrument for investigating bound state energy and its mass. In these instances, actual relativity-based revisions are minimal, and the theoretical challenge is reduced to obtaining relativity-based revisions for the nonrelativistic interaction potential based on Feynman's functional path integral and within quantum field theory and quantum electrodynamic ideas [13, 14]. In this article, we try to describe the inter-hadronic potential with relativity-based revisions through the nonrelativistic limit. This method studies primarily focused on relativistic corrections within perturbation theory for the interaction potential. It is well known that calculating the total binding energy and wave functions of bound states consisting of multiple particles from the relativistic Schrödinger equation is nearly impossible from a mathematical standpoint. Hence, we define incorporating Einsteinian adjustments into the determination of relativistic bound state properties using Feynman's functional path integral and quantum field theory formalism. As we know, the bound state mass with the required quantum numbers of the respective currents is determined by the asymptotic behavior and limiting properties of the correlation function [15]. The correlation function, described in terms of the Green function and is represented as a functional Feynman path integral in nonrelativistic quantum mechanics, which enables the necessary asymptotic behavior of wave function at large distances, and accurately performs the averaging and integrating over the external field. In this case, the mass spectrum of coupled state in the radial Schrödinger equation is the constituent mass which differs from the mass of the initial state of the coupled system, and also the resulting exchange of the gauge field is determined by the Feynman diagram that we know as interaction potential. Therefore, as a result, the constituent mass of the particles can explain and present the relativistic corrections to the total Hamiltonian. Therefore, we start with the issue of the flow of charged particles in the arbitrary background field $F_B(r)$ with $U_B(r)$the Bernoulli potential type and present Green's function $G_m\,(r,r'|F_B)$ of filed. In quantum field theory, Green's function (also known as a propagator) is a mathematical tool used to describe the behavior of particles or fields in terms of correlations between different points ($r\to r'$) in spacetime. Green's function helps us to understand how a disturbance in one part of the system propagates through spacetime to affect other parts of the system. In quantum mechanics and quantum field theory Green's function $G_m\,(r,r'|F_B)$ is a solution to the Schrödinger equations for a given field, subject to specific boundary conditions [16]

$$[(i\partial_r+gF_B(r))^2+m^2]G_m\,(r|F_B)=\delta(r-r'). \quad (12)$$

Here, r and r' are spacetime points and g is a coupling constant. As we know Green's functions are typically used to compute the correlation functions, which are the basic building blocks for calculating scattering amplitudes, cross-sections, and interaction properties in particle physics. Computing Green's functions for specific fields and interactions is a crucial step in understanding the behavior of particles and fields in quantum field theory and in making predictions for high-energy physics experiments. Therefore, we present the behavior of particles in the external field based on the current and Green's function. The current of a scalar charged particle is $J(r) = R^{+}(r)R^{-}(r)$, and then

$$J(r) = -(-i)e\int d^3r' G_{m_1}(r,r'|F_B)\nabla'\varphi'(r'), \quad (13)$$

where e is the charge of particles and $\varphi(r)$ is a scalar Gaussian potential field. The scalar product current of two scalar charged particles is

$$\langle J(r)J(r')\rangle = \langle R^{+}(r)R^{-}(r)\cdot R^{+}(r')R^{-}(r')\rangle, \quad (14)$$

is equivalent to $\langle G_{m_i}(r|F_B)G_{m_j}(r'|F_B)\rangle$,

where

$$G_{m_i,m_j}(r_i,r_j;r'_i,r'_j) = \langle 0|\hat{T}\varphi(r_i)\varphi(r_j)\varphi(r'_i)\varphi(r'_j)|0\rangle, \quad (15)$$

is the propagator or kernel function of a scalar-charged particle with mass m_i in the arbitrary external field, $\hat{T}$ is the time-ordered product of operators. The Green's function in the conventional form can determine the correlators by averaging over the arbitrary external field $F_B(r)$ and determine the loop function (polarization function) as follows

$$\Pi(r-r') = \langle J(r)J(r')\rangle = \langle G_{m_i}(r|F_B)G_{m_j}(r'|F_B)\rangle. \quad (16)$$

Hence, the polarization function of two scalar particles with masses m_i, m_j reads

$$\Pi(r-r') = \langle G_{m_i}(r,r'|F_B)\cdot G_{m_j}(r,r'|F_B)\rangle = -2i\sum\int\frac{d^3r}{(2\pi)^3}G_{m_i}(r,r'|F_B)\cdot G_{m_j}(r\,,r'|F_B). \quad (17)$$

Now, one can formulate a variational method that will be used in determination of the Green's function and the polarization function based on Feynman's functional path integral form and define them as follows [13,16]; one can propose the functional integral

$$G(r,r'|F_B) = \int d\sigma e^{-gU[\varphi]},$$

where

$$d\sigma = \frac{1}{N}\delta\varphi e^{\{-0.5\int\int drdr'\varphi(r)G^{-1}(r,r')\varphi(r')\}}, \quad (18)$$

where $\delta\varphi$ is the functional differential. N is the normalization scaling coefficient which can be defined from the normalization condition $\int d\sigma = 1$, G^{-1} is the differential operator. $G\,(r,r'|F_B)$ is the Green's function and define by

$$\int dr' G^{-1}(r,r')G\,(r',r) = \delta(r-r')$$

and then

$$G(r,r'|F_B) = \frac{1}{\breve{N}}\delta\varphi e^{\{-0.5\int dr(\varphi(r))^2 - gU[\varphi]\}}. \quad (19)$$

With the condition $G(0) = 1$ we can define $\breve{N}$ as a constant parameter. After some algebraic representation with the variational parameters q,s the functional integral $G(r,r'|F_B)$ is presented in the following way

$$G(r,r'|F_B) \geq e^{\{M(g)\}}. \quad (20)$$

We know, the coupled system's mass spectrum should explain in relativistic quantum theory as relations $M = -\lim_{|r|\to\infty}|r|\ln\Pi(r)$, where $\Pi(r) = (\langle G_1(r)|G_2(r)\rangle)$[12], so the upper estimation for $M(g)$ is $M(g) \leq M_{+}(g)$:

$$M_{+}(g) = min\left\{-0.5\sum_n\left[ln(1+q_n) - \frac{q_n}{1+q_n}\right] - 0.5\sum_n s^2 - \int d\sigma U[\varphi,s]\right\}. \quad (20)$$

Based on $M(g)$ one can define that the quadratic form of functional $M(g)$ gives us exact and complete results. Now the Green's function in the 4D space reads

$$G_m\,(r,r'|F_B) = \int_0^\infty \frac{d\mu}{(4\pi\mu)^2} e^{\left\{-\mu m^2 - \frac{(r-r')^2}{4\mu}\right\}} \times \int d\sigma_\xi e^{\left\{ig\int_0^1 d\xi \frac{dZ(\xi)}{d\xi} F_B(\xi)\right\}}, \quad (22)$$

and,

$$Z\,(\xi) = (r-r')\xi + r' - 2\sqrt{\mu}B(\xi),$$

$$d\sigma_\xi = N\delta B(\xi) e^{\left\{-0.5\int_0^1 d\xi\,(B(\xi))^2\right\}},$$

Where N is the normalization condition where $\int d\sigma_\xi = 1$ with boundary conditions $B(0) = B(1) = 0$. Using the above presentation of functional $G(r,r'|F_B)$ for the Green's function and the polarization function we define [12]

$$G_m\,(r,r'|F_B) = \langle G_m\,(r,r'|F_B)\rangle = \int_0^\infty \frac{d\mu}{(4\pi\mu)^2} e^{\left\{-\mu m^2 - \frac{(r-r')^2}{4\mu}\right\}} . \, e^{-M_+((r-r'),\mu)}, \quad (23)$$

$$M_+ = \min_{(\xi,\sigma,\lambda)>0}\left\{\frac{1}{4\xi} + 2\sigma + \frac{\lambda}{2} + g\xi\int_0^\infty d\mu f(k)\right\}, \quad (24)$$

where

$$f(k) = \int e^{-\mu}\int\left(\frac{dk}{2\pi}\right)^4 \breve{D}(k^2)\left[1 - e^{\left\{\frac{ikn\mu}{2\lambda} - \frac{\xi}{6}k^2\left(1-e^{-\frac{\sigma\mu}{\lambda}}\right)\right\}}\right],$$

and

$$\Pi(r-r') = \int\int_0^\infty \frac{d\mu_i d\mu_j}{(8\pi^2\mu_i\mu_j|r-r'|)^2}\,\Omega(\mu_i,\mu_j) \times e^{\left\{-\frac{|r-r'|}{2}\left[\left(\frac{m_i^2}{\mu_i}+\mu_i\right)+\left(\frac{m_j^2}{\mu_j}+\mu_j\right)\right]\right\}}, \quad (25)$$

where

$$\Omega(\mu_i,\mu_j) = c_i c_j \int\int \delta\mu_i\,\delta\mu_j e^{\left\{-\int_0^\alpha d\tau \sum_{k=1}^2 0.5\mu_k \dot{F_B}^2 - U_{ij}\right\}}, \quad (26)$$

$$U_{ij} = -U_{ii} + 2U_{ij} - U_{jj}\;, \quad (27)$$

We reduce 4D space of the functional $\Omega(\mu_i,\mu_j)$ to the 3D space (i.e., we neglect the time component) due to calculate integrals before describing the functional $\Omega(\mu_i,\mu_j)$ as resembling the behavior of two particles with masses μ_i, μ_j in the nonrelativistic quantum mechanics and with the potential and nonpotential interaction that includes in U_{ij} in the form of the Feynman path integral. Hence, based on the above presentation of the propagator (the Green's function) at the limited distant $|r-r'| \to 0$ we can determine the coupled state total mass M_+ of two bounded particles with the rest masses m_i, m_j as follows

$$G_m\,(r,r'|F_B) = \langle G_m\,(r,r'|F_B)\rangle \geqslant \frac{const}{|r-r'|} e^{\{-M_+|r-r'|\}}, \quad (28)$$

where

$$M_+ = min\left\{\frac{1}{4\xi} + 2\sigma + \frac{\lambda}{2} + g\xi\int_0^\infty d\mu f(k)\right\}, \quad (29)$$

which depends on the coupling constant and correlation function. $M_+ = g^{0.5}\left(\int\left(\frac{dk}{2\pi}\right)^4 \breve{D}(k^2)\right)^{0.5}$ for the weak interaction $g \ll 1$ and the strong interaction $g \gg 1$ satisfy the relation

$$M_+ = (1.022g)^{0.25}\left(\int\left(\frac{dk}{2\pi}\right)^4 \breve{D}(k^2)\,k^2\right)^{0.25}, \quad (30)$$

and also, the functional $\Omega(\mu_i,\mu_j)$ which contains potential and nonpotential interactions at the asymptotic behavior $|r-r'| \to 0$ is equivalent to the relation

$$\Omega(\mu_i,\mu_j) \cong e^{\{-|r-r'|E_{n\ell}(\mu_i,\mu_j)\}}, \quad (31)$$

where the function $E_{n\ell}(\mu_i,\mu_j)$ directly depends on μ_i, μ_j, m_i, m_j, g and it is an eigenvalue of the Schrödinger equation $HR(r-r') = E_{n\ell}(\mu_i,\mu_j)R(r-r')$ of two bounded particles with the Hamiltonian: $H = \frac{1}{2\mu_i}p_i^2 + \frac{1}{2\mu_j}p_2^2 + U_{ij}(|r-r'|)$. Parameters m_i, m_j are the current masses of interacting particles, μ_i, μ_j are the constituent masses of interacting particles i.e., the mass of particles inside the coupled state.

Now considering the above equations and relations, we define the polarization function for two bounded particles, at the asymptotic limit $|r-r'| \to 0$:$\Pi(r-r') \cong e^{\{-M_+|r-r'|\}}$, therefore the bound state mass spectrum is defined as

$$M_{+} = -\lim_{|r-r'|\to\infty} \frac{1}{|r-r'|} \ln\Pi(r-r'). \tag{32}$$

The mass of bound state determined analogs for the propagator function in Feynman's functional path integral form based on the method of exponential asymptotic or the method of steepest descent looks like

$$M_{+} = \frac{\partial}{\partial\mu_{i,j}}\left(\left(\frac{\mu_j m_i^2 + \mu_i m_j^2}{2\mu_i\mu_j}\right) + 0.5(\mu_i+\mu_j) + E_{n\ell}(\mu_i,\mu_j)\right) =$$

$$\min_{\mu_i,\mu_j}\left(\left(\frac{\mu_j m_i^2 + \mu_i m_j^2}{2\mu_i\mu_j}\right) + 0.5(\mu_i+\mu_j) + E_{n\ell}(\mu_i,\mu_j)\right). \tag{33}$$

Therefore, we can define

$$\mu_i^2 + 2\mu_i^2\frac{\partial}{\partial\mu_i}E_{n\ell}(\mu_i,\mu_j) - m_i^2 = 0, \tag{34}$$

$$\mu_j^2 + 2\mu_j^2\frac{\partial}{\partial\mu_j}E_{n\ell}(\mu_i,\mu_j) - m_j^2 = 0. \tag{35}$$

From section 2, we can define the Green's function of two intertwined transformed spaces as follows

$$\left(\frac{d^2}{dr^2} - \frac{L(L+1)}{r^2} + W\right)G(r-r') = \frac{2\mu}{\hbar^2}\delta(r-r'),$$

$$\left(\frac{d^2}{dq^2} - \frac{L_D(L_D+1)}{q} + \breve{W}\right)G(q-q') = \frac{2\breve{\mu}}{\hbar^2}\delta(q-q'). \tag{36}$$

Now we present the radial Schrödinger equation with relativistic energy formula based on the quantum field theory description in Eq. (33) and use approximation relation ($c = \hbar = 1$)

$$E = (p^2 + m_0^2)^{1/2} \cong \frac{1}{2}\min_{\mu_i}\left(\mu_i + \frac{p^2+m_0^2}{\mu_i}\right). \tag{37}$$

Therefore, $HR = E_{n\ell}R$ the radial Schrodinger equation for bounded particles with rest masses m_i,m_j and the constituent masses μ_i,μ_j reads [12]

$$\left[\frac{1}{2}\left(\mu_i + \frac{p^2+m_i^2}{\mu_i}\right) + \frac{1}{2}\left(\mu_i + \frac{p^2+m_j^2}{\mu_j}\right) + U - -M\right]R(r) = 0, \tag{38}$$

or

$$\left[\frac{p^2}{2\mu} + \mu U\right]R = \left[M - \frac{\mu_i+\mu_j}{2} - \frac{m_i^2\mu_{j+}m_j^2\mu_i}{2\mu_i\mu_j}\right]R, \tag{39}$$

where

$$E_{n\ell}(\mu_i,\mu_j) = M - \frac{\mu_i+\mu_j}{2} - \frac{m_i^2\mu_j + m_j^2\mu_i}{2\mu_i\mu_j}, \tag{40}$$

$$\frac{1}{\mu} = \frac{1}{\mu_i} + \frac{1}{\mu_j}, \tag{41}$$

μ is reduced mass, μ_i, μ_j present the mass of particles in the bound state, in other words, it is a relativistic mass of particle

$$\mu_i = \frac{m_i c^2}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{42}$$

m_i is the rest mass and using equations (34) and (35), we can define the constituent mass of a particle as follows

$$\mu_i = \sqrt{m_i^2 - 2\mu^2\frac{\partial}{\partial\mu_i}E_{n\ell}(\mu_i,\mu_j)}. \tag{43}$$

6 Mass spectrum of upsilon meson

The radial Schrödinger equation (Eq. (39)) of the upsilon meson system with the Bernoulli interaction's potential type built on $\left(U_H(r) = -A\frac{e^{-m_D r}}{1-e^{-m_D r}}\right)$ Hulthen potential reads

$$\left\{\frac{p^2}{2\mu} + U_B(r) - E_{n\ell}(\mu)\right\}R(r) = 0, \tag{44}$$

where m_D is the Debye mass and $E_{n\ell}(\mu) = E_{n\ell}(\mu_i,\mu_j)$ and Hulthen potential based on Eq. (3) has the form

$$U_H(r) = -\frac{Ae^{-m_D r}}{1-e^{-m_D r}} \to -\frac{A}{e^{m_D r}(1-e^{-m_D r})} = -\frac{A}{e^{m_D r}-1} \to -\frac{A}{r}\frac{r}{e^{m_D r}-1} = -\frac{A}{r} f_B(x)$$

where $f_B(x)$ is Bernoulli series $f_B(x) = B_0 + B_1\frac{x}{1!} + B_2\frac{x^2}{2!} + B_4\frac{x^4}{4!} + B_6\frac{x^6}{6!} + \cdots$. Hence, we define Hulthen's potential up to the fourth order by the Bernoulli series

$$U_B(r) = \frac{A}{2} - \frac{A}{m_D}\frac{1}{r} - \frac{Am_D}{2.3!}r + \frac{Am_D^3}{30.4!}r^3. \qquad (45)$$

Then Eq. (44) using two intertwined spaces transformation (Eq. (10)) $\hat{r} = \hat{q}^{2\rho},\ R(r) \to \mathfrak{R}(\hat{q}^{2\rho})$ takes the form

$$\varepsilon_0(E_{n\ell})\mathfrak{R} = \left\{\frac{\hat{p}_q{}^2}{2\mu} + 4\rho^2\hat{q}^{4\rho-2}\left(U_B - E_{n\ell}(\mu)\right)\right\}\mathfrak{R} = 0. \qquad (46)$$

The upsilon meson bound state wave function becomes an oscillator. Now, we will analytically calculate the mass spectrum and energy eigenvalue of Eq. (46) using the quantum oscillating properties condition of the bound state with the Hamiltonian $H = H_0 + H_I$, where H_0 is the Hamiltonian of free oscillators and H_I is the Hamiltonian of interactions (or it is directly related to the perturbation of the system). The bound state of the quantum oscillating system can be presented by the Wick ordering method in the Symplectic Space (WOSS). It is formulating the canonical variables in terms of raising $\hat{a}^+$ and lowering $\hat{a}$ operators in the D-dimension space i.e.,

$$\hat{q} = \left[\frac{2m\omega}{\hbar}\right]^{1/2}(\hat{a}^+ + \hat{a}),$$
$$\hat{p}_q = i\left[\frac{m\omega}{2\hbar}\right]^{1/2}(\hat{a}^+ - \hat{a}), \qquad (47)$$

where ω is the oscillator frequency. Substituting the canonical variables $\hat{q}, \hat{p}_q$ into the equation (46) and ordering by the creation and annihilation operators, the interaction Hamiltonian is obtained as follows [12]

$$H = \omega(\hat{a}^+\hat{a}) + \frac{D}{2}\omega + \int\left(\frac{dk}{2\pi}\right)^D \breve{W}(k^2)e^{\frac{-k^2}{4\omega}} :e^{ik\hat{q}}: - \frac{\omega^2}{2}\left(:\hat{q}^2: + \frac{D}{2\omega}\right). \qquad (48)$$

As we know WOSS method requires that the H_I would not contain the quadratic form of the normal ordering of operators $:\hat{q}^2:, \hat{q}^2.$, these terms are included in the H_0. Based on this condition one can determine ω by relation $\omega^2 + \int\left(\frac{dk}{2\pi}\right)^D\left(\frac{k^2}{D}\right)e^{\frac{-k^2}{4\omega}}\breve{W}(k^2) = 0$. The canonical variables in Eq. (46) using Eq. (47) read

$$\hat{p}_q^{2u} = \omega^u\frac{\Gamma(\frac{D}{2}+u)}{\Gamma(\frac{D}{2})} + :\hat{p}_q^2: \omega^{u-1}u\frac{\Gamma(\frac{D}{2}+u)}{\Gamma(\frac{D}{2}+1)} + :\aleph: \approx \omega^u\frac{\Gamma\left(\frac{D}{2}+u\right)}{\Gamma\left(\frac{D}{2}\right)},$$

$$\hat{q}^{2u} = \frac{1}{\omega^u}\frac{\Gamma(\frac{D}{2}+u)}{\Gamma(\frac{D}{2})} + :\hat{q}^2: \frac{u}{\omega^{u-1}}\frac{\Gamma(\frac{D}{2}+u)}{\Gamma(\frac{D}{2}+1)} + :\aleph: \approx \frac{1}{\omega^u}\frac{\Gamma(\frac{D}{2}+u)}{\Gamma(\frac{D}{2})}$$

and then Eq. (46) reads

$$\varepsilon_0(E_{n\ell}) = \frac{D\omega}{4} + 4\rho^2\hat{q}^{4\rho-2}\left(\frac{A}{2} - \frac{A}{m_D}\frac{1}{\hat{q}^{2\rho}} - \frac{Am_D}{2.3!}\hat{q}^{2\rho} + \frac{Am_D^3}{30.4!}\hat{q}^{6\rho} - E_{n\ell}(\mu)\right)\mathfrak{R} = 0. \qquad (49)$$

Using a series of mathematical transformations and relations, we determine ω, $E_{n\ell}(\mu)$, M and the mass parameters μ, μ_i, μ_j by applying the main condition of the WOSS method in the creation of a bound state at the minimum of oscillator frequency and energy eigenvalue. Therefore, $\frac{d\varepsilon_0(E_{n\ell})}{d\rho} = 0$ and $\frac{d\varepsilon_0(E_{n\ell})}{d\omega} = 0$. The former enables to define $\rho = 1$, while the latter enables determining ω as follows

$$\omega^5 - \frac{8\mu a}{D}\omega^4 + 2\mu c(D+2)\omega^2 - \frac{6\mu g}{D} = 0, \qquad (50)$$

where

$$a = \frac{A}{m_D}, \quad b=\frac{A}{2}, \quad c=\frac{Am_D}{12}, \quad f=\frac{Am_D^3}{30.4!},$$
$$g=f\frac{D(D+2)(D+4)(D+6)}{4},$$

$$\omega_{min} = \frac{64\mu A}{15Dm_D}. \tag{51}$$

The energy eigenvalue is defined by integrating the two equations $\varepsilon_0(E_{n\ell}) = 0$ and $\frac{d\varepsilon_0(E_{n\ell})}{d\omega} = 0$, so it reads

$$E_{n\ell}(\omega,\mu) = \frac{\omega^2}{8\mu} - \frac{2a\omega}{D} + \frac{g\omega^{-3}}{2D} - \frac{c(D+2)\omega^{-1}}{2} + b. \tag{52}$$

Equation (43) determines the constituent mass parameters μ_i, μ_j using. We should have to find the reduced mass value, one can determine it using relation (43)

$$\frac{1}{\chi} = \frac{1}{\sqrt{1+\beta^2\chi^2}} + \frac{1}{\sqrt{y^2+\beta^2\chi^2}}, \tag{53}$$

where $\chi = \frac{m_i}{\mu}$, $\beta = \left(\frac{64a}{15D}\right)^2$. Thus, the mass spectrum of the upsilon meson coupled state is determined and Eq. (53) gives the relativistic mass of particles based on transformed nonrelativistic Schrödinger equation to the relativistic form using properties of quantum filed theory representation in the path integral form.

7 Numerical and theoretical data

In the previous section, we computed the mass spectrum of upsilon meson for the ground state and radially excited states in the Bernoulli potential based on the Hulthen potential. In the next calculation, we suppose the bound state creation depends on finite temperature and is introduced in the equations via Deby mass. The Deby mass parameter of potential is fitted to experimental data from extracted from reference [17]. Then solving Eqs. (50),(52), and (53), which are obtained from the WOSS method, we utilize the numerical values of the upsilon meson bound state, which has the quark's rest mass $m_i = m_j = 4.823\ GeV$[17]. We determine the finite temperature effect using well known relation $m_D = 14.652\alpha_s(T)T$. In the numerical calculations $\alpha_s(T) = 2\pi/(11 - \frac{2N_f}{3})log(0.057T)$, $N_f = 3$. We take the temperature range $(30 < T < 150)$MeV, and choose the criminal temperature $T_c = (170 \pm 16)MeV$ [18, 19].

Table 1. Mass spectrum and constituent mass of upsilon meson in at $m_D = 1.520GeV$, m_i =4.823 GeV, $A = -1.591GeV$. (all values are in (GeV)).

ℓ	0	1	2
ω	0.986	1.544	1.933
μ	2.566	2.447	2.427
μ_i	4.848.	4.884	4.918
M	9.469	10.003	10.233
M_{exp} [17]	9.460	9.899	10.164
M_{teor}[22]	9.459	9.618	10.256
M_{teor} [22]	9.460	9.619	9.864
M_{teor} [20]	9.477	9.900	9.862
M_{teor} [21]	9.510	10.155	10.214

The upsilon meson mass spectrum M and relativistic mass of quarks $\mu_i = \mu_j$ in the ground and excited states and bound state frequency are computed numerically in Table 1. Then the upsilon meson ground state probability density at finite temperatures with the relativistic mass corrections, in the zero-point energy state are presented in Fig. 1.

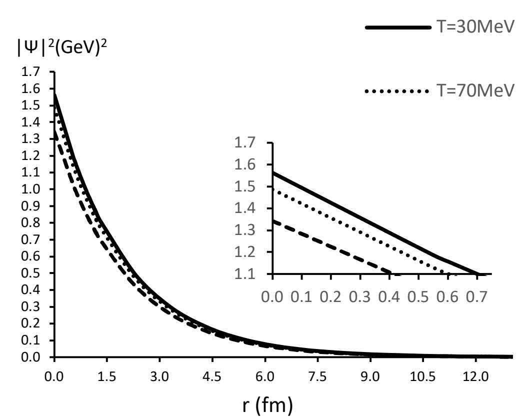

Figure 1. The upsilon meson ground state probability density at the finite temperatures ($T = (30, 70, 90)MeV$.

8 Conclusions

In this study, we utilized Bernoulli's potential to calculate the upsilon meson-bound state. We used the transformation of two Symplectic intertwined spaces and defined all operators in the form of the Wick ordering method. Then based on QFT and the Feynman path integral properties of the system, the approximation form of $\sqrt{p^2+m_0^2} \cong \frac{1}{2}\min_{\mu_i}\left(\mu_i + \frac{p^2+m_0^2}{\mu_i}\right)$ is obtained. The approximate solutions to the quantized energy values and the constituent mass of quarks in the bound states by modifying the radial Schrödinger equation are defined. These results can be employed to compute the masses of heavy and light mesons, all hadronic bound states including ${D_s}^+, {B_c}^+, B^+, D^+, D^{*+}$ in the ground and excited states. We described the upsilon-bound state properties taking into account relativistic conditions and the Debye mass in the finite temperatures with the strong interaction. We applied the WOSS method to obtain an analytical solution for the radial Schrödinger equation. The calculated masses are found to be in good agreement with experimental data as well as with the work of other researchers. As we showed, this simple nonrelativistic potential model is useful for approximating the properties and characteristics of bound states within a thermal environment and at finite temperature. Other potential models, such as the Bernoulli type employed in this paper, can still be analyzed and compared. In this study, we have already solved the radial Schrödinger equation, with the relativistic corrections which is a crucial step in the correlator calculation based on QFT. The systematic approach employed in this paper stands as one of the most specific works in this field and has the potential to be significant in many branches of physics, particularly in hadronic, nuclear, and atomic physics. The main and important points of this article are 1- Obtaining the approximate form of the potential with high-order exponents and removing even or odd exponents according to Bernoulli expansion. 2- Obtaining the relation of relativistic energy and its approximation for the Schrödinger equation considering the introduction of the relativistic mass of particles that is not present in the Schrödinger equation. 3- Simplifying the approximate investigation of particle interactions at high temperatures.

Conflicts of interest

The authors have no conflicts of interest.

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