Canonical thermostatics of the ideal gas in the framework of the generalized uncertainty principle

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ABSTRACT

In this study, The statistical consequences of minimal length supposition are investigated for a canonical ensemble of ideal gas. These effects are encoded in the so-called Generalized Uncertainty Principle (GUP) of the second order. In the framework of the considered GUP scenario, a unique partition function is obtained by using two different methods of quantum and classical approaches. It should be noticed that here we consider the magnitude of the momentum in the deformed Hamiltonian of the model. In this way, the model is different from the already existing model which does not have any significant result in the quantum approach. In particular, the corrections to the thermodynamical characteristics such as the mean energy, the entropy and the density of states are achieved. The induced improvements manifest themselves at very high-temperature limits. However, it is shown that, if one applies the predicted observational bound on the GUP deformation parameter, the modifications become more observable even at intermediate temperatures. The deformation parameter of the considered GUP model is also estimated for nowadays precision of measurements of the heat capacity of an ensemble of hydrogen atoms.

1 Introduction

One of the principal challenges of modern physics today is to conciliate the concepts of quantum gravity and ordinary quantum mechanics. One of the topics of these notions is the possible minimum scale of length which is contemplated by quantum gravity theories. This is an ancient conception which has attracted much attention during the history of science. Recently, this problem also finds a new concern by the progress of quantum gravity theories such as string theories and semi-classical black hole physics [1]-[4], loop quantum gravity [5]-[6], and doubly special relativity [7]-[8]. However, in order to incorporate the concept of the minimal scale of length into ordinary quantum mechanics, the Heisenberg uncertainty principle,

should be modified. Thus, the Generalized Uncertainty Principle (GUP) has been established.

With the growth of research in GUP models, it has become more necessary to guarantee the existence of the minimal scale of length and reveal its characteristics and hence observe and examine the outgoings of GUP models. Statistical mechanics provides a framework to examine the quantum results of a physical system and hence can prepare a powerful tool to test GUP modifications of quantum mechanics as well. In the past few years, many papers have appeared in the literature to study the statistical consequences of the presence of a minimum measurable length in the context of the GUP model. Some of these researches are devoted to the

thermodynamics for the FRW universe [9] and the early universe [10], the black holes [11]-[12], the gravity's Rainbow [13]-[14]. The thermodynamics of the relativistic ideal gas in the GUP framework is analyzed in [15]. With the minimal length uncertainty relation, the Maxwell-Boltzmann statistics is investigated [16]. The density of state deformation and an improved definition of the statistical entropy in a GUP model have been introduced in [17] and [18].

Recently we have introduced a simple method to investigate the statistical consequences of GUP models [19] and [20]. The procedure is based on the perturbation theory. In fact, the GUP induced relative corrections of different physical quantities are small in such a way that the perturbation method works well. Moreover, in GUP models, only the leading terms of the modifications parameters appearing in the Heisenberg uncertainty principle are kept. Hence, the Gup models themselves have perturbation expansion [21].

In our previous work, we have investigated the first order influence of GUP on the thermodynamics of the canonical ensemble of the ideal gas. We have applied our method to find the statistical characteristics of a canonical ensemble of distinguishable ideal gas and harmonic oscillator systems. The proposed procedure is efficient and is deficient from the usual difficulties. The modified partition function of the system has been obtained by applying the first order perturbation method. The associated thermodynamical quantities of the system such as the mean energy, entropy, Helmholtz free energy, the chemical potential, and specific heat and pressure have been achieved by using the obtained partition function. It is observed that in the framework of the considered GUP model, the mean energy, the specific heat, and entropy are reduced as compared with their ordinary values. However, some other quantities such as the Helmholtz free energy and chemical potential increase, while the pressure remains unchanged. The density of states has also been obtained in which the GUP induced term acquires a negative value proportional to the mass of the particles.

These investigations have been applied to the GUP model of the first order of the deformation parameter. In the present task, we consider the second order GUP model [6]. This version of GUP is more

comprehensive and besides being consistent with the concept of the so-called minimal length scale, provides the concept of the maximal momentum scale. In the framework of this version of GUP, we study the modifications of the thermodynamics of the canonical ensemble of the ideal gas.

The GUP model, we apply here, is presented by the commutator:

$$[x,p]=i\hbar\left(1-2\alpha p+4\alpha^2 p^2\right),\qquad(1)$$

where $\alpha > 0$ is the deformation parameter so that for $\alpha = 0$, the ordinary commutator relation is recovered. The parameter α is given as $\alpha=\alpha_0\frac{L_{Pl}}{\hbar}=\frac{\alpha_0}{M_{Pl}c}$ with L_{Pl} and M_{Pl} being the Planck length and mass, respectively and α_0 is a free parameter. Nowadays, the best upper bound on the magnitude of α_0 can be found from the Lamb shift experiment precision in hydrogen as $\alpha_0 < 10^{10}$.

The modified commutator (Eq. (1)) leads to the following generalized uncertainty principle:

$$\Delta x\Delta p\geq\frac{\hbar}{2}\left(1-2\alpha\langle p\rangle+4\alpha^2<p^2>\right).\qquad(2)$$

The generalized position x and momentum p, which satisfy the commutator relation (Eq. (1), in position representation, can be written as follows:

$$x = x_0,\ p = p_0(1-\alpha p_0+2\alpha^2 p_0^2).\qquad(3)$$

where, x_0 and p_0 satisfy the canonical commutation relation $[x_0,p_0]=i\hbar$. Here we consider p_0 as the magnitude of the momentum operator. This is different from the already existing supposition.

Using Eq. (3), it has been shown that any non-relativistic Hamiltonian of the form $H=\frac{p^2}{2m}+V(\vec{r})$ can be written as:

$$H = H_0 + \Delta H,\qquad(4)$$

where H_0 is the ordinary Hamiltonian:

$$H_0 = \frac{p_0^2}{2m} + V(\vec{r}), \tag{5}$$

where ΔH denotes its deviation due to the GUP effects:

$$\Delta H = H_1 + H_2 + H_3 + H_4, \tag{6}$$

where

$$H_1 = -\frac{\alpha}{m} p_0^3, \qquad H_2 = \frac{5\alpha^2}{2m} p_0^4,$$

$$\tag{7}$$

$$H_3 = -\frac{2\alpha^3}{m} p_0^5, \qquad H_4 \frac{2\alpha^4}{m} p_0^6,$$

where the Hamiltonians H_i represent the perturbations.

This article attempts to study the consequences of the statistical mechanics of an ensemble of the ideal gas in the context of the GUP model of Eq. (1). The ensemble is supposed to be composed of systems consisting in N identical and noninteracting molecules, confined to a space of volume V. The number N is normally assumed to be extremely large as the order of 10^{23}. It is considered a canonical ensemble in which the system is supposed to be immersed in a very large heat reservoir. On attaining a state of mutual equilibrium, the system of the ideal gas and the reservoir would have a common temperature T. The partition function of the system is obtained by using two different methods, namely quantum and classical approaches. Then, the modified statistical quantities of the system are obtained in a straightforward manner.

This paper is organized as follows: In section two the modified spectrum of a particle confined in a well is obtained in the framework of the considered GUP model. In section three by applying the quantum approach, the modified partition function of a canonical ensemble of monatomic ideal gas is calculated up to the first order of the deformation parameter of the model. In section four, by employing the phase space mechanism it is attempted to find the partition function of the same system. As expected, the two approaches lead to the same partition function.

Section five is devoted to specifying the improved thermodynamical characteristics of the system by using the obtained partition function. Finally, section six presents the results.

2 Modified spectra of a particle in a well

Let us suppose a system that consists of a particle of mass m confined to a one-dimensional square well of width L. The potential energy of the particle is given by:

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 \le x \le L \\ \infty & L < x \end{cases} \tag{8}$$

In order to estimate the Planck scale effect on the energy spectrum of the particle, we apply here the standard perturbation theorem. The full Hamiltonian is given by Eq. (4) where the term ΔH should be considered as its perturbation part. The allowed energy spectrum of the bound state solutions for the unperturbed Hamiltonian is given by

$$E_n^{(0)} = \frac{\pi^2 \hbar^2}{2mL^2} n^2, \, n = 1,2,3,,... \tag{9}$$

In the framework of the standard perturbation theory, the explicit expansion of the energy shift of the nth level takes the following form:

$$\Delta_n = E_n - E_n^{(0)} = (\Delta H)_{nn} + \sum_{k \ne n} \frac{|(\Delta H)_{kn}|^2}{E_n^{(0)} - E_k^{(0)}} + ... \tag{10}$$

where $(\Delta H)_{kn}$ are the matrix elements of ΔH with respect to the unperturbed eigenvectors, namely

$$(\Delta H)_{kn} = \langle k^{(0)} | \Delta H | n^{(0)} \rangle. \tag{11}$$

In the well, $V(x) = 0$ and the unperturbed Hamiltonian is $H_0 = \frac{p_0^2}{2m}$.

The unperturbed Hamiltonian H_0 is degenerate and for any magnitude p_0 we have two different eigenvectors $|n^{(0)}\rangle = |p_0\rangle$ and $|n^{(0)}\rangle = |-p_0\rangle$. The perturbed Hamiltonians H_i's, via relation Eq. (7), are

functions of p_0 which we consider here as the magnitude of the momentum operator. Hence, when we apply H_i's on them, both $|n^{(0)}\rangle$ result in positive values. Hence for odd powers of p_i's in H_i's we do not obtain zero for the elements $(H_i)_{nn}$ in the integral summations due to the oddness of the integrand. However, if one does not consider p_0 as the magnitude of the momentum operator, then the elements $(H_i)_{nn}$ obtain positive and negative values which cancel each other in the summation. This already existing case does not contain any significant consequences and the leading terms of the modified energy become zero. Instead, by regarding p_0 in the modified Hamiltonians H_i's, as the magnitude of momentum, the modified energy levels become significant.

Using Eq.(10), the correction to the energy spectrum is then given as:

$$\Delta_n = (\Delta H)_{nn}, \tag{12}$$

also $\Delta H_{kn} = 0$ for $k \neq n$. Using Eqs. (6) and (7), the energy shift can be decomposed as:

$$\Delta_n = \Delta_{n1} + \Delta_{n2} + \Delta_{n3} + \Delta_{n4}, \tag{13}$$

where

$$\Delta_{ni} = (H_i)_{nn}. \tag{14}$$

The energy shift Δ_{n1} then can be calculated as:

$$\Delta_{n1} = (H_1)_{nn} = -\frac{\alpha}{m}\langle n^{(0)} | p_0^3 | n^{(0)}\rangle,$$

$$= -\frac{\alpha}{m}\left(2mE_n^{(0)}\right)^{\frac{3}{2}}. \tag{15}$$

The other parts of the energy shift can similarly be obtained as follows:

$$\Delta_{n2} = \frac{5\alpha^2}{2m}\left(2mE_n^{(0)}\right)^2,$$

$$\Delta_{n3} = -\frac{2\alpha^3}{m}\left(2mE_n^{(0)}\right)^{\frac{5}{2}},$$

$$\Delta_{n4} = \frac{2\alpha^4}{m}(2mE_n^{(0)})^3, \tag{16}$$

Equations (15) and (16) present the GUP corrections to the energy spectrum of a particle in the well (Eq. (8)). The strength of these modifications depends on the mass of the particle and the energy values $E_n^{(0)}$. The Planck scale effects are therefore enhanced for massive particles and upper energy levels. Let us now examine the validity of our perturbation method and the convergence of the energy series. Using Eq. (15), one obtains:

$$\frac{\Delta_{n1}}{E_n^{(0)}} = -\frac{2\alpha_0}{M_{Pl}c}(2mE_n^{(0)})^{1/2}. \tag{17}$$

For the hydrogen atom with $m = 511\,keV/c^2 = 9.11\times10^{-31}\,kg$ and $L = 0.2\,nm$ (the diameter of the atom), the energy spectrum, from Eq.(9), becomes $E_n^{(0)} \approx 1.51\times10^{-18}\,n^2\,J$. Then, for $M_{Pl} = 2.1\times10^{-8}\,kg$, Eq. (17) yields:

$$\frac{\Delta_{n1}}{E_n^{(0)}} = -0.53\times10^{-24}\,\alpha_0\,n. \tag{18}$$

Equation (15) shows that for the upper limit $\alpha_0 < 10^{10}$ and the energy levels $n < 10^{14}$, the conditions $\frac{\Delta_n^{(1)}}{E_n^{(0)}} < 1$ is well satisfied. Similarly, we have:

$$\frac{\Delta_{n(i+1)}}{\Delta_{ni}} \propto -\frac{\alpha_0}{M_{Pl}c}(2mE_n^{(0)})^{1/2}, i = 1,..4 \tag{19}$$

which, yeilds:

$$\frac{\Delta_{n(i+1)}}{\Delta_{ni}} \propto -0.264\times10^{-24}\,\alpha_0\,n. \tag{20}$$

Equation (20) again yields the condition $|\frac{\Delta_n^{(i+1)}}{\Delta_n^{(i)}}| < 1$ for $\alpha_0 < 10^{10}$ and $n < 10^{14}$. Therefore, the energy series expansion converges and the applied perturbation method is meaningful.

3 Modified quantum canonical partition function

Consider a canonical ensemble of ideal gas composed of systems that each consists of N identical noninteracting particles enclosed by an adiabatic wall with width L. Suppose that the systems are kept at equilibrium by being in contact with a heat bath at temperature T. The macrostate of the ensemble is defined through the parameters N, L, and T. The partition function of a single molecule of a system is given by:

$$Q = \sum_n e^{-\beta E_n}, \tag{21}$$

where $\beta = \dfrac{1}{kT}$ and k is the Boltzmann factor. Considering $E_n = E_n^{(0)} + \Delta_n$, Eq. (21) can be rewritten as:

$$Q = \sum_n e^{-\beta\left(E_n^{(0)} + \Delta_n\right)}. \tag{22}$$

Applying a simple series expansion to Eq.(22), allow the following form for the modified partition function:

$$Q = \sum_n (1 + \Delta_{n1} + \Delta_{n2} + \Delta_{n3} + \Delta_{n4} + ...)e^{-\beta E_n^{(0)}}. \tag{23}$$

In view of the largeness of the number of states of a particle and the largeness of the width of the well to which the particle is confined, we convert the sum into an integral, treating n as a continuous variable, namely $\sum \rightarrow \int dn$, [23]. In this way, the first term of Eq. (23) yields the unperturbed partition function:

$$Q^{(0)} = \int_0^\infty dn\, e^{-\beta E_n^{(0)}} = \frac{L}{\hbar}\left(\frac{m}{2\pi\beta}\right)^{\frac{1}{2}} \tag{24}$$

where, $E_n^{(0)} = \dfrac{\pi^2\hbar^2}{2mL^2}n^2$ has been replaced. The corrections to the partition function can then be found by the relation:

$$\Delta Q = \int_0^\infty dn\, (\Delta_{n1} + \Delta_{n2} + \Delta_{n3} + \Delta_{n4} + ...)e^{-\beta\frac{\pi^2\hbar^2}{2mL^2}n^2}. \tag{25}$$

Substituting Eqs. (15) and (16), the integration of Eq. (25) leads to:

$$\Delta Q = \alpha \frac{2m}{\beta}\frac{L}{\pi\hbar}\left[1 - 10\alpha^2\frac{2m}{\beta} - 30\alpha^3\frac{2m}{\beta}\left(\frac{2m\pi}{\beta}\right)^{\frac{1}{2}}\right]. \tag{26}$$

Equation (26) relates the partition function induced by the considered GUP model. The complete quantum partition function for a one-particle one-dimensional system in the framework of the considered GUP model then is given by:

$$Q = Q^{(0)} + \Delta Q. \tag{27}$$

4 Modified classical canonical partition function

For classical systems, the most appropriate framework for developing the formalism of specifying the thermodynamics is the phase space [24, 25]. In classical mechanics, a state of motion of N particles in one dimension is uniquely determined by the N coordinates and N momentums. Each pair (q_i, p_i), corresponds to one point in a $2N$-dimensional space in the phase space. Consider a canonical ensemble of ideal gas composed of systems where each consist of N identical noninteracting particles enclosed by the adiabatic well with width L. Suppose that the systems are kept at equilibrium by being in contact with a heat bath at temperature T. The macrostate of the ensemble is defined through the parameters N, L and T. For a system in a heat bath it is sufficient to calculate the partition function that yields the free energy, from which follows all properties of the system at a given temperature.

The partition function in the phase space, for a single particle in a system of the ensemble, is given by:

$$Q = \frac{1}{h}\int dp_0\, dq\, e^{-\beta H}, \tag{28}$$

where $dp_0\, dq$ is the differential element of the phase space. It is supposed that the particle has no interaction with its surroundings so that the energy of the particle is wholly kinetic. The corresponding Hamiltonian is given by Eq. (4) and substituting in Eq. (28), leads to the partition function:

$$Q = \frac{1}{h}\int dp_0\, dq\, e^{-\beta(H+\Delta H)}, \tag{29}$$

In order to compute Eq. (29), a simple series expansion to the integrand $e^{-\beta \Delta H}$ is first applied. Then, by rearranging terms with respect to the parameter α, one finds:

$$Q = \frac{L}{h}\int_{-\infty}^{\infty} dp_0 \begin{bmatrix} 1 - \alpha\beta H_1 + \alpha^2\left(-\beta H_2 + \frac{\beta^2}{2}H_1^2\right) \\ +\alpha^3\left(-\beta H_3 + \beta^2 H_1 H_2 - \frac{\beta^3}{3!}H_1^3\right) \\ +\alpha^4\begin{pmatrix} -\beta H_4 + \frac{\beta^2}{2}H_2^2 + \beta^2 H_1 H_3 \\ -\frac{\beta^2}{2}H_1^2 H_2 - \frac{\beta^4}{4!}H_1^4 \end{pmatrix} \end{bmatrix} \tag{30}$$

where $\int dq = L$. The first term of Eq. (30) yields the ordinary classical partition function as:

$$Q_0 = \frac{L}{h}\int_{-\infty}^{\infty} dp_0 e^{-\frac{\beta p_0^2}{2m}} = \frac{L}{\hbar}\left(\frac{m}{2\pi\beta}\right)^{\frac{1}{2}}, \tag{31}$$

which is equal to the quantum one, namely Eq. (24). Inserting the definitions H_i's from Eq. (7), integrating the other terms of Eq. (30) yields the same relation as Eq. (26). This means that the classical and quantum approaches results in the same partition function for the considered system in the framework of the considered GUP model.

Relation (27) states the one-dimensional partition function of a one particle system. The N-particle partition function in three dimensions then can be obtained by using the relation $Q_{3N} = \frac{(Q)^{3N}}{N!}$. This relation is correct by the assumption of homogeneity and isotropy of the space-time and the indistinguishability of the particles.

5 Modified thermodynamics

The modified partition function of the system, in the presence of the GUP model, has been calculated in the last section. Using the partition function, let us now obtain some statistical characteristics of the considered canonical system. The mean energy of the system in the ensemble, namely $U = \frac{E}{V}$, can be found by using the relation:

$$U = -\frac{\partial}{\partial\beta}\ln(Q_{3N}). \tag{32}$$

Using Eq. (27), we have:

$$\ln Q_{3N} = N + N\ln\left[\frac{V}{N}\left(\frac{m}{2\pi\beta\hbar^2}\right)^{3/2}\right] + 6N\alpha\left(\frac{2m}{\pi\beta}\right)^{1/2}$$
$$- 120N\alpha^3\left(\frac{m}{\beta}\right)\left(\frac{2m}{\pi\beta}\right)^{\frac{1}{2}}$$
$$- 720N\alpha^4\left(\frac{m}{\beta}\right)^2, \tag{33}$$

where considering $\frac{m}{\beta} < 1$, the series expansion of the form $\ln(1+x); x - x^2/2$ is applied. Substituting Eq. (33) in Eq. (32), the mean energy can be found in the following form:

$$U = \frac{3}{2}NkT\left[1 + 2\alpha\left(\frac{2m}{\pi\beta}\right)^{1/2} - 120\alpha^3\left(\frac{m}{\beta}\right)\left(\frac{2m}{\pi\beta}\right)^{\frac{1}{2}}\right.$$
$$\left. - 960\alpha^4\left(\frac{m}{\beta}\right)^2\right], \tag{34}$$

Here the first term is the ordinary mean energy in the absence of GUP while the other terms show the GUP induced corrections. The GUP modification terms depend on the particle number N, the mass of each particle m, the temperature T and the deformation parameter α. From Eq. (34) it is obvious that the energy increases in the presence of the applied GUP model.

The specific heat at constant volume follows from:

$$C_V = \left(\frac{\partial U}{\partial T}\right)_{N,V}, \tag{35}$$

where by substituting in Eq. (34), leads to:

$$C_V \approx \frac{3}{2}Nk\left[1 + 3\alpha\left(\frac{2m}{\pi\beta}\right)^{\frac{1}{2}} - 300\alpha^3\left(\frac{m}{\beta}\right)\left(\frac{2m}{\pi\beta}\right)^{\frac{1}{2}}\right.$$
$$\left. - 2880\alpha^4\left(\frac{m}{\beta}\right)^2\right]. \tag{36}$$

The first term $\frac{3}{2}Nk$ is the ordinary specific heat at constant volume. The other terms present the GUP modifications due to the minimal length suppositions. In the absence of the GUP modification, namely for $\alpha = 0$, the ordinary specific heat is obtained. Equation (36) shows that the specific heat increases in the framework of the GUP model.

Another important quantity is the Helmholtz free energy that is given by:

$$A(V,T) = -kT\, ln\,(Q_{3N}). \tag{37}$$

By substituting in Eq. (33), the modified Helmholtz free energy can be obtained. Then by using the result for the modified Helmholtz free energy, in the entropy relation

$$S = -\left(\frac{\partial A}{\partial T}\right)_{N,V}, \tag{38}$$

one obtains:

$$S = Nk\left\{\frac{5}{2} + +\ln\left[\frac{V}{N}\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2}\right] + 9\alpha\left(\frac{2m}{\pi\beta}\right)^{1/2}\right.$$
$$- 300\alpha^3\left(\frac{m}{\beta}\right)\left(\frac{2m}{\pi\beta}\right)^{\frac{1}{2}}$$
$$\left. - 2160\alpha^4\left(\frac{m}{\beta}\right)^2\right\}. \tag{39}$$

Equation (39) shows the modified entropy in the presence of the GUP model. The first two terms show the ordinary entropy in the absence of GUP ($\alpha = 0$), while the other terms relate to the corrections. Hence in the framework of the GUP model, the total entropy of the system increases.

The pressure of the system is given by:

$$P = -\left(\frac{\partial A}{\partial V}\right)_{N,T}, \tag{40}$$

whiere by substituting in Eq. (37) gives the familiar ideal gas law,

$$P = \frac{NkT}{V}, \tag{41}$$

and hence the considered GUP model has no effect on the pressure of the system which is consistent with the result of [19].

Another important thermodynamic characteristics are the density of states. The further expression for the partition function is given by:

$$Q_{3N} = \int_0^\infty e^{-\beta E}g(E)dE, \tag{42}$$

where $g(E)$ denotes the number of states around the energy value E, [24]. This relation indicates that for $\beta > 0$, the partition function Q_{3N} is the Laplace transform of $g(E)$. Hence using a simple extension of inverse Laplace transform one can immediately deduce that:

$$g(E) = \frac{1}{2\pi i}\int_{\beta'+i\infty}^{\beta'-i\infty}e^{\beta E}Q_{3N}(\beta)\,d\beta \quad \beta' > 0 \tag{43}$$

Substituting Q_{3N} from Eq. (27) in Eq. (43) and using the following formula:

$$I = \frac{1}{2\pi i}\int_{\beta'+i\infty}^{\beta'-i\infty}\frac{e^{\beta x}}{\beta^{n+1}}d\beta = \frac{x^n}{n!}, \tag{44}$$

for $x \geq 0$, and $I = 0$ for $x \leq 0$, leads to

$$g(E) \approx 3Ng_0(E)\left[\frac{1}{3N} + 2\alpha\left(\frac{2mE}{\pi}\right)^{1/2} + \frac{12}{\pi}\alpha^2NmE\right.$$
$$+ 12N^2\alpha^3\left(\frac{2mE}{\pi}\right)^{3/2}$$
$$\left. - \frac{480}{\pi}N\alpha^4m^2E^2\right], \tag{45}$$

for $E \geq 0$ and $g(E) = 0$ for $E \leq 0$. Here, $g_0(E)$ is the ordinary state density in the absence of the considered GUP mode which is given by:

$$g_0(E) = \frac{V^N}{N!}\left(\frac{m}{2\pi\hbar^2}\right)^{\frac{3N}{2}} \frac{E^{\frac{3N}{2}-1}}{(\frac{3N}{2}-1)!}. \qquad (46)$$

In computing Eq. (45), the supposition of the largeness of N is applied. At the limit $\alpha = 0$, from Eq. (45) one obtains $g(E) = g_0(E)$. In Eq. (45), the terms containing the parameter α are the corrections in the presence of the minimal length effects. It is noticeable here that the number of state increases in the presence of the considered GUP model.

6 Upper bound on the modification parameter

Let us now estimate the modifications of the thermodynamical characteristics obtained in the last section. By a priori assumption, the deformation parameter $\boldsymbol{\alpha_0}$ is of the order of $\boldsymbol{\alpha_0 \approx 1}$ which induces a very small amount to the calculated thermodynamical characteristics. However, one may assign upper limits, higher than unity, to the magnitude of $\boldsymbol{\alpha_0}$ to made the modifications more measurable. For example, consider Eq. (36) which gives the specific heat formula in the presence of the considered GUP model. The relative correction with respect to the ordinary specific heat is:

$$\frac{\Delta C_V}{C_V^{(0)}} \approx 3\alpha \left(\frac{2m}{\pi\beta}\right)^{\frac{1}{2}} - 300\alpha^3 \left(\frac{m}{\beta}\right)\left(\frac{2m}{\pi\beta}\right)^{\frac{1}{2}}$$

$$- 2880\alpha^4 \left(\frac{m}{\beta}\right)^2, \qquad (47)$$

This modification , for hydrogen atoms $(m = 9.11 \times 10^{-31} kg)$ at room temperature (T=300 K), and for the Planck mass $(M_{Pi} = 2.18 \times 10^{-8} kg)$ can be estimated as follows:

$$\frac{\Delta C_V}{C_V^{(0)}} \approx 0.22 \times 10^{-25}\alpha_0 - 0.198 \times 10^{-75}\alpha_0^3 - 0.22$$

$$\times 10^{-100}\alpha, \qquad (48)$$

Today's accuracy on the experimental values of C_V can be considered about 10^{-7}, [26]. This is the accuracy of the experimental values of the universal gas constant

obtained by measuring the speed of sound, which can be related to the value of C_V. Considering this precision, from the first term of Eq. (48), the following upper bound can be set on the magnitude of the deformation parameter α_0:

$$\alpha_0 < 10^{18}. \qquad (49)$$

This upper bound is near to the value set by the position measurements ($\alpha_0 < 10^{17}$), [27, 28]. However, this upper bound is weaker than that set by the hydrogen Lamb shift effect ($\alpha_0 < 10^{10}$) and the electron tunneling effect ($\alpha_0 < 10^{11}$) [22].

7 Conclusions

This study has found the thermodynamical consequences of the Planck scale of length and momentum on an ensemble of the ideal gas. In the framework of the considered version of the GUP relation, these scales manifest themselves in the Hamiltonian of the system, as two terms. The modified partition function of the system has been obtained by using two different methods, quantum, and classical approaches. These procedures lead to an identical result for the modified partition function.

The mean energy, the specific heat, and entropy are increased as compared with their values in the HUP scheme, while the pressure remains unchanged. The density of states have also been obtained in which the GUP induced term acquire an overall positive value proportional to the mass of the particles. The consequences of this paper, about the reduction or increase of the obtained quantities, conflict with the results already been obtained in our previous work [19]. These inconsistencies are reasonable and appear due to applying different GUP models. In comparison, the leading terms of the modified Hamiltonian in these two GUP models have different signs. In fact, the first correction in the Hamiltonian we applied here, is of the form $H_1 = -\frac{\alpha}{m} p_0^3$ which is negative. However the modified Hamiltonian in

[19] is $\dfrac{\alpha}{m} p_0^4$ which is positive. consequently, for example, the mean energy increases here but decreases in [19].

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