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A Numerical Study of KPZ Equation Based on Changing its Parameters

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ABSTRACT

In this article we investigate the behavior of the scaling exponents of the KPZ equation through changing three parameters of the equation. In other words we would like to know how the growth exponent β and the roughness exponent α will change if the surface tension ν, the average velocity λ and the noise strength Υ change. Using the discrete form of the equation, first we come to the results $\alpha = 0{:}5$ and $\beta = 0{:}33$, then we change the parameters in the range of 0 to 3 by small amounts; we observe that the exponents change smoothly. In some limit states the KPZ equation transforms to the RD and RDSR equations. Fortunately this is highlighted in the figures. As the parameters change and approach the limit states, the KPZ universality class evolves to the RD and RDSR regimes.

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چکیده

در این مقاله ما در مورد رفتار نما های مقیاس معادله KPZ از طریق تغییرات پارامتر های آن بحث می کنیم . به بیان دیگر مایلیم بدانیم نمای رشد β و نمای زبری α چگونه با تغییر دادن تنش سطحی ν ، سرعت متوسط λ و شدت نویز γ تغییر می کنند. با استفاده از شکل گسسته معادله ابتدا به نتایج β= ۰/۳۳ و α =۰/۵ می رسیم ، سپس پارامتر ها را در محدوده ۰ تا ۳ به مقدار کم تغییر می دهیم ، مشاهده می کنیم که نما ها به آرامی تغییر می کنند . در حالت های حدی معادله KPZ به معادله های RD و RDSR تبدیل می شود . خوشبختانه این موضوع در نمودار ها دیده می شود . وقتی که پارامتر ها تغییر می کنند و به حالت های حدی نزدیک می شوند کلاس جهانی معادله KPZ به رژیم های RD و RDSR تبدیل می شود.

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1 Introduction

The KPZ equation is a non-linear stochastic partial differential equation which was formulated by Mehran Kardar , Giorgio Parisi, and Yi-Cheng Zhang in 1986. It describes the temporal change of height $h(x,t)$ at location x and time. In one dimension it is formally written:

$$\frac{\partial h(x,t)}{\partial t} = \nu\nabla^2 h(x,t) + \frac{\lambda}{2}\left(\nabla h(x,t)\right)^2 + \eta(x,t), \quad (1)$$

where $\eta(x\ ,\ t)$ is a Gaussian white noise with average $\langle\eta(x,t)\rangle = 0$ and a second moment expressed as:

$$\langle\eta(x,t)\,\eta(x',t')\rangle = 2\,\gamma\,\delta(x-x')\delta(t-t'), \quad (2)$$

Where ν , λ, and γ are the parameters of the model [5].

The KPZ equation is known to describe nonequilibrium models such as BD, EDEN, SOS, etc. but it is not restricted to the subject of surface growth. In fact it has become a paradigm to govern a vast class of nonequilibrium phenomena. This equation has a close relationship with the Burgers equation [2] which is an important equation in the context of turbulence. The KPZ equation also maps to the directed polymers problem. It transforms to diffusion equation by means of Hopf-Cole transformation whose field can be interpreted as the partition function of the specific path of the directed polymers [1]. It has been shown that the formation of large scale structures from clusters in the universe to galaxy distribution is governed by the KPZ equation [9]. The roughening of flame fronts occurs in the KPZ equation universality class [3]. An important property of the KPZ equation is that besides temporal and special symmetries, it follows the Galilean invariance [4].

Moving at a constant velocity v will not affect any physical law of the system such as equation of motion. This means that under these transformations the KPZ equation remains invariant :

$$x \to x - \lambda v t \quad , \quad h \to h + v x, \quad (3)$$

It is equal to tilting the interface by infinitesimal angle, which lead to the result:

$$\alpha + z = 2, \quad (4)$$

which is valid in any dimension, where z is the dynamic exponent defined as:

$$z = \frac{\alpha}{\beta}. \quad (5)$$

Of course in one case the Galilean invariance is broken and different scaling exponents result [6]. We can obtain the roughness exponent using the fluctuation- dissipation theorem in one dimension [1]. This theorem states that a change or fluctuation in the system will be dissipated as the system returns to equilibrium [8]. This can be seen in the stationary probability distribution:

$$P_{st} \sim \exp\left(-\frac{\nu}{2\,\gamma}\int dx\,(\partial_x h)^2\right), \quad (6)$$

which is exactly the stationary probability distribution for Edwards-Wilkinson equation. The roughness exponent of the EW equation is $\alpha = 0.5$. So when the system in the KPZ model saturates and reaches the stationary state, it will behave like a system in the EW model, where the roughness exponent of the KPZ equation is $\alpha = 0.5$.

Many studies have been carried out regarding this equation, e.g. changing the angle of deposition of particles [10], working out the crossover in the equation due to different initial conditions [11], and highlighting the statistical viewpoint in its regard [12].

2 Solution of KPZ equation

Since the KPZ equation is nonlinear, it does not have any analytical solution. A common way of treating this equation is to linearize it by means of the Hopf-Cole transformation [4].

$$h(x,t) = \frac{2\nu}{\lambda}\ln\left(\varphi(x,t)\right). \quad (7)$$

Applying it to the KPZ equation, reaction – diffusion equation (diffusion equation with multiplicative noise) results

$$\frac{\partial \varphi(x,t)}{\partial t} = \nu \nabla^2 \varphi(x,t) + \varphi(x,t)\eta(x,t), \qquad (8)$$

which attributes to the directed polymers problem [1]. The purpose of solving this equation is to find the scaling exponents. This means that any system following this model will have the same scaling exponents. The surface that grows by the EW and KPZ models are known to be self-affine. Self-affine surfaces are supposed to remain invariant under rescaling:

$$x \to x' = bx, \qquad h \to h' = b^{\alpha} h\,, t \to t' = b^{z} t, \qquad (9)$$

In contrast to the EW equation we cannot find the scaling exponents of the KPZ equation by rescaling it. By rescaling the EW equation the parameters ν , λ, and γ remain constant, but by rescaling the KPZ equation those parameters change.

The standard method of solving a nonlinear stochastic partial differential equation is the renormalization group approach [7]. In this method we consider the nonlinear term as the perturbative term and expand the equation in the Fourier space. Finally we can obtain the flow equations for ν , λ where by setting equal to zero the exponents can be found [1].

2 Numerical solution of the KPZ equation

In order to solve the KPZ equation we use numerical methods. We write the discrete form of the equation [4]. Since the KPZ equation is a stochastic equation, in order to avoid errors due to noise fluctuations in our calculations, to obtain the discrete form we apply the inverse Hopf-Cole transformation expressed as

$$\varphi_j(t) = \exp\left(\frac{\lambda}{2\nu} h_j(x,t)\right), \qquad (10)$$

to the diffusion equation (with multiplicative noise).

$$\dot{\varphi}_j = \frac{\nu}{a^2}\left(\varphi_{j+1} - 2\varphi_j + \varphi_{j-1}\right) + \frac{\lambda\sqrt{\gamma}}{2\nu}\varphi_j \eta_j, \qquad (11)$$

so we get the discrete form of the KPZ equation.

$$h_j(t) = \frac{\nu}{a^2} + \frac{\lambda}{4a^2}\left[\left(h_{j+1} - h_j\right)^2 + \left(h_j - h_{j-1}\right)^2\right] + \sqrt{\gamma}\eta_j. \qquad (12)$$

We choose a surface of zero height to be grown in the KPZ model. It is supposed to have 100 sites. The time interval of a deposition is 0.01 (the time taken for the surface to grow each site from the beginning to the end). We repeat this action 1500 times (total time). The height of a particular site is calculated by:

$$h_j(t+1) = h_j(t) + \Delta t\left[\frac{\nu}{a^2}\left(h_{j+1}(t) - 2h_j(t) + h_{j-1}(t)\right) + \frac{\lambda}{4a^2}\left[\left(h_{j+1}(t) - h_j(t)\right)^2 + \left(h_j(t) - h_{j-1}(t)\right)^2\right] + \sqrt{\gamma}\eta_j(t)\right]. \qquad (13)$$

The interface width is defined as:

$$w(L,t) = \sqrt{\frac{1}{L}\sum_{i=1}^{L}\left(h(i,t) - \bar{h}(t)\right)^2}, \qquad (14)$$

where $\bar{h}(t)$ is the mean height:

$$\bar{h}(t) = \frac{1}{L}\sum_{i=1}^{L} h(i,t). \qquad (15)$$

The relation between w and β is:

$$w(L,t) \sim t^{\beta}. \qquad (16)$$

We plot the logarithm of interface width as a function of logarithm of time, so β is the slope of the fitting line before saturation (Fig. 1). In Fig. 1 we demonstrate the growing part of the process in green so we do not consider the saturation part to calculate the growth exponent. The red line is the fitting line of the growing part. The slope of this line is 0.29. The value of β for the

KPZ equation is 0.33. The amount of error is 0.04.

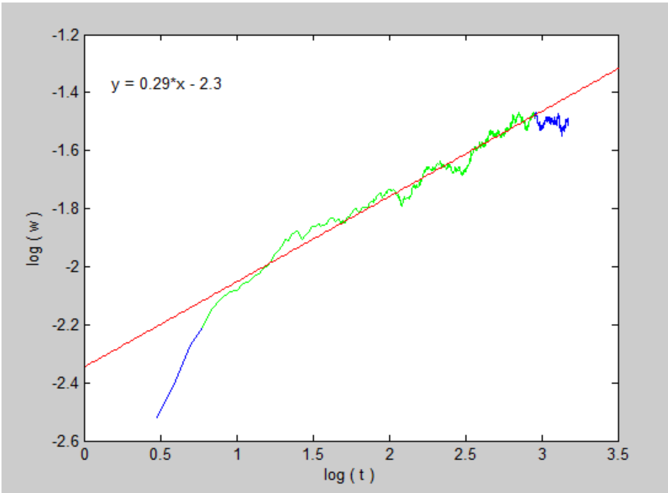

Fig.1: growth exponent of the KPZ equation

The relation between interface width and roughness exponent is:

$$w_{sat}(L) \sim L^{\alpha}, \qquad (17)$$

where w_{sat} is the saturation width. If the interface width is plotted as a function of system size (L) in saturation time, α is the slope of the fitting line. A better way of calculating the roughness exponent is to divide the surface to small parts known as windows. We calculate the interface width for each window and then average them .Then we increase the size of the windows and again we calculate the interface width for each window and average them. This process is repeated as many times as the interface fits 4 or 5 windows. At last we plot the interface of every step as the size of the windows in every step.

In Fig. 2 we did so and the graph is logarithm of interface width as the logarithm of system size (the blue graph) .Then we plotted the fitting line of the graph whose slope is the value of α. The fitting line in Fig. 2 is in red and its slope is 0.45. The value of α for the KPZ equation is 0.5. The amount of error is 0.05.

Now we change the amount of λ and γ to see what happens to the exponents. In figure 3 we demonstrate a cubic (three dimensional) plot of β with three axes (every one representing a parameter). In the figures the values of ν and λ change from 0 to 3.9 and the value of γ changes from 0.05 to 0.5. It could be noticed that the amount of β decreases along the λ axis from 0.5 up to about 0.3. Clearly it shows the change in regime from the RD to KPZ , but there is no change in β along the γ axis, we come to conclusion that changing the noise strength does not affect the growth exponent.

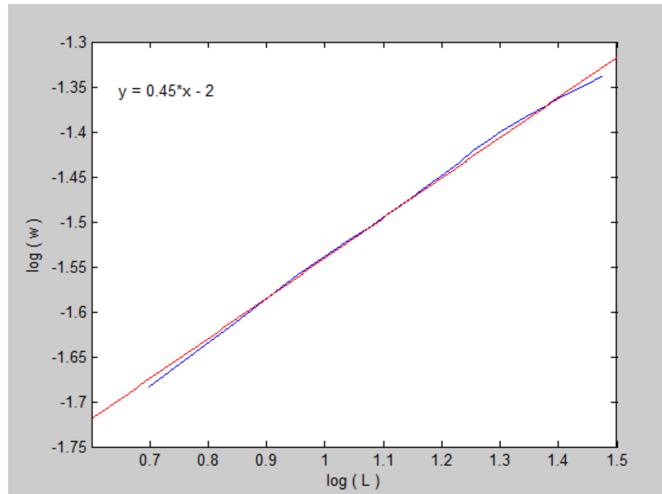

Fig.2: roughness exponent of the KPZ equation

We expect to observe a regime change along the ν axis from the RD to RDSR as the amount of ν changes from 0 to 3.9. But little change in β along the ν axis occurred, and after running the program for many times we can see that the RD regime tends to evolve to the RDSR regime.

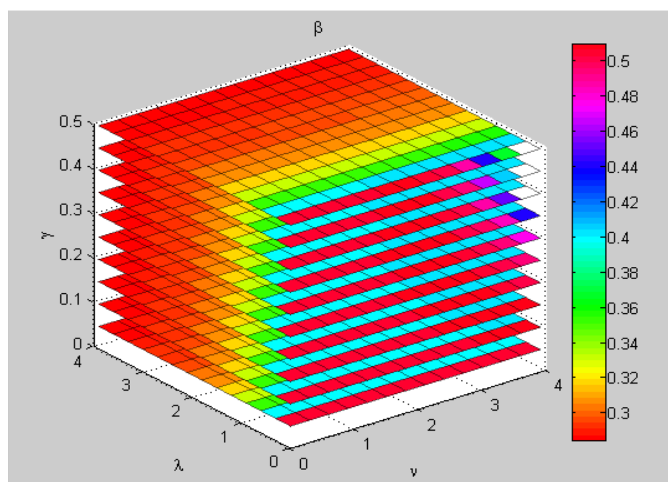

Fig. 3: Cubic plot for the growth exponent of the KPZ equation

To better understand the cubic plot we take the axes apart and analyze them one by one. Figure 4 is the plot of β only as a function of, so the values of λ = 0 and γ = 0.2 are constant. It is clear that the value of β varies from 0.5 to 0.35 and the system tends to evolve from the RD to RDSR regime, although it did not reach the amount of 0.25 which is the value of beta for the RDSR regime.

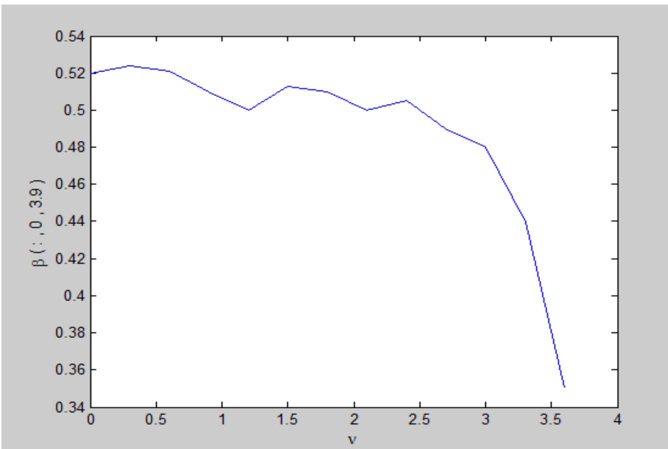

Fig.4: The variations of β with respect to ν

Again we refer to Fig. 3 to see the variations of β with respect to λ and ν. Figure 5 is the plot of β as a function of λ, while the values of ν = 0 and γ = 0.3 are constant. Note that the value of β varies from 0.5 to about 0.3 and the system evolves from RD to KPZ regime.

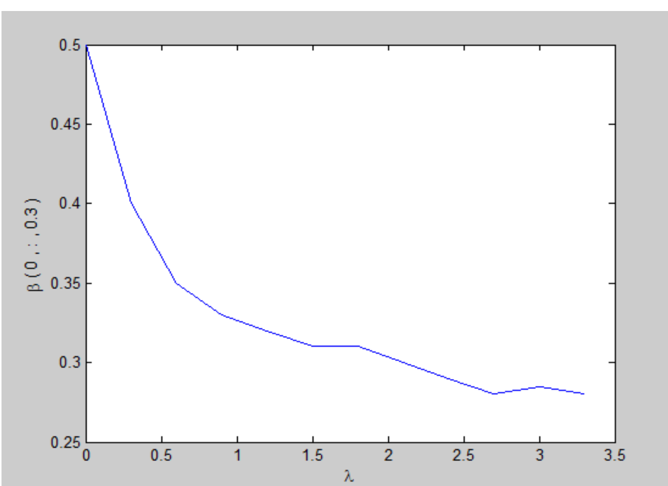

Fig. 5: The variations of **β** with respect to **λ**

In Fig. 6 we considered ν = 0 and λ = 1.5, so β is a function of γ. As we observed in Fig. 3 there is approximately no change in β by changing γ.

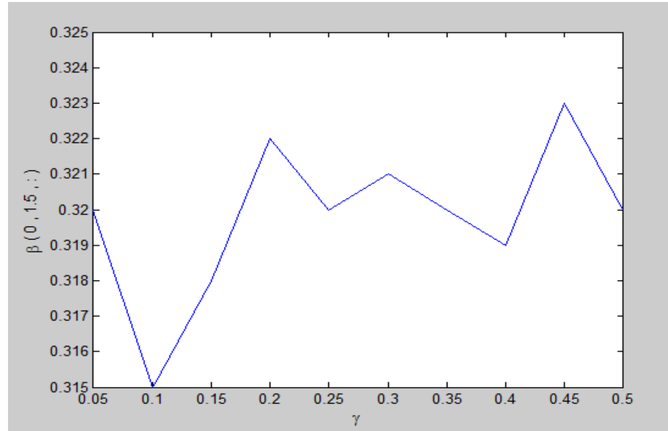

Fig. 6: The variations of β with respect to γ

We do the same action for roughness exponent. Again the points λ = 0 and ν = 0 represent the RD regime with α = 0. By moving along the ν axis we gradually go to RDSR regime with α = 0.5 and by moving along the λ axis we go to KPZ regime with α = 0.5 . This is shown in Fig. 7.

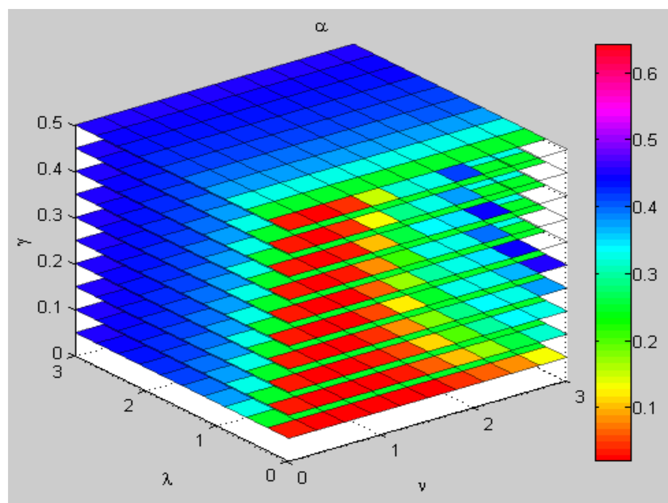

Fig. 7: Cubic plot of roughness exponent of KPZ equation

Now we take the axes apart to observe the α behavior individually. In Fig. 8 we consider λ = 0 and γ=2.5, it is clear that the regime changes from RD (= 0) to RDSR (α =0.5) .

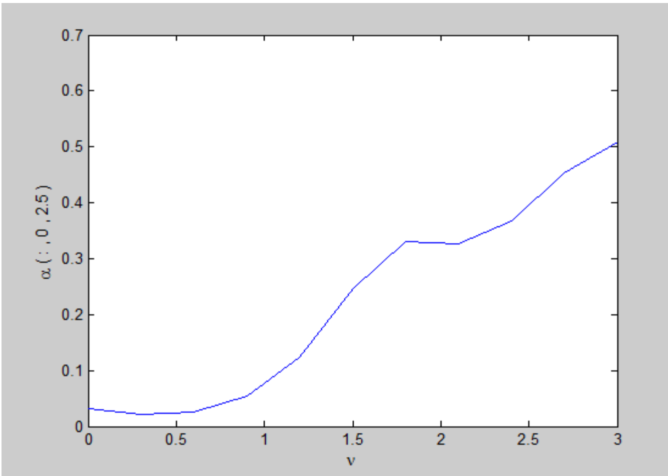

Fig. 8: Change of α by changing ν

In Fig. 9 we consider $\nu = 0$ and $\gamma = 0.45$, we see that by increasing λ from 0 to 3, α increases from 0 (RD regime) to 0.5 (KPZ regime). In a research on the KPZ equation it was shown analytically that in weak coupling regime the crossover occurs [13]. The coupling constant is defined as:

$$g = \frac{\gamma\lambda^2}{\nu^3}. \tag{18}$$

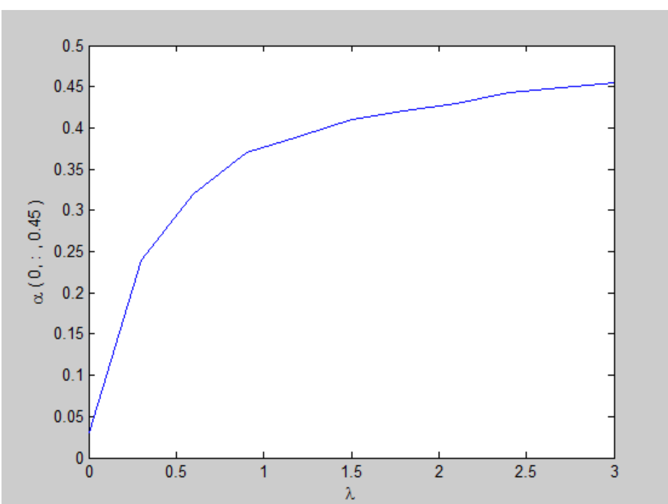

Fig. 9: Change of α by changing λ

In Fig. 6 we observed that changing γ has no effect on, now we see the same for α. In Fig. 10 we consider $\nu = 0.6$ and $\lambda = 0.9$, as we see there is almost no change in α.

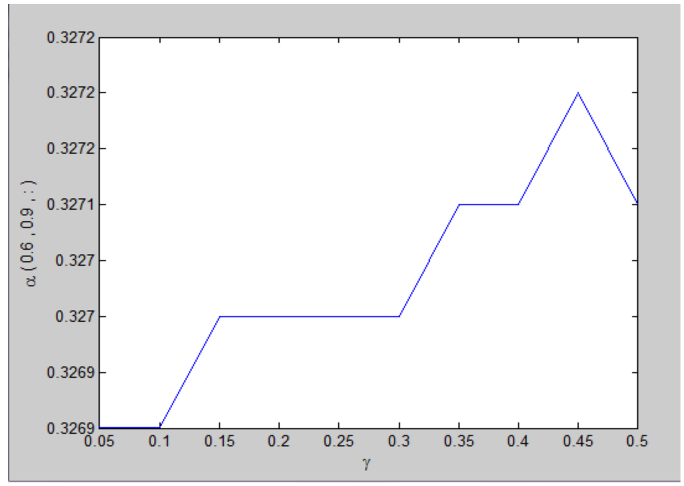

Fig. 10: Change of α by changing γ

The weak coupling regime is equivalent to small coupling constant (small λ or large) which is EW equation. We showed this crossover from KPZ equation to EW equation numerically.

4 Conclusions

In this article we concentrated on scaling exponents of the KPZ equation, a fundamental equation in complex systems. We reviewed some properties of this equation and then focused on its solution. As we know the purpose of solving the equation is to find the scaling exponents. The scaling exponents of the equation are known (= 0.33 and $\alpha = 0.5$). Our work was to change the values of the parameters of the equation to see if there is any difference in the value of each exponent, the answer was positive. Further going we showed that the change in the KPZ regime to the RD and RDSR regimes occurred. We concluded that changing noise strength has the least effect on the change of α and β exponents, while changing ν and λ are effective in changing α and β, so that it leads to change of the KPZ regime.

Acknowledgments

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